Section 6.3

Tuesday, April 14, 2020 7:34 PM

Inverse trigonometric substitution

The arcsine substitution: Suppose that we see the term $\sqrt{a^2-x^2}$, where $a>0$, in our integrand. (OR, a^2-x^2)

Then we can try to use the substitution

$\theta = \arcsin\left(\frac{x}{a}\right)$ OR $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = |a\cos\theta| = a\cos\theta$$

$a>0$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

Example: Find $\int \frac{1}{x\sqrt{9-x^2}}\,dx$

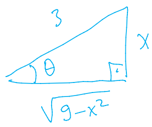
Solution:

$$\int \frac{1}{x\sqrt{9-x^2}}\,dx = \int \frac{\cancel{3\cos\theta}\, d\theta}{3\sin\theta\ \cancel{3\cos\theta}} = \frac{1}{3}\int \csc\theta\, d\theta$$

$$= \frac{1}{3}\left(-\ln|\csc\theta+\cot\theta|\right)+C$$

$$= \frac{-1}{3}\ln\left|\frac{3}{x}+\frac{\sqrt{9-x^2}}{x}\right|+C$$

$$\boxed{\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\, d\theta \\ \sqrt{9-x^2} &= 3\cos\theta \end{aligned}}$$



The arctangent substitution: Suppose that we have $\sqrt{a^2+x^2}$, where $a>0$, in our integrand. (OR, a^2+x^2)

Then we can try to use the substitution

$\theta = \arctan\left(\frac{x}{a}\right)$ OR $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = |a\sec\theta| = a\sec\theta$$

$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

Example: Find $\int_0^1 \frac{x^2}{(4+x^2)^2}\,dx$

Solution:

$$\int_0^1 \frac{x^2\,dx}{(4+x^2)^2} = \int_0^{\arctan(\frac{1}{2})} \frac{\cancel{4}\tan^2\theta\ \cancel{2}\cancel{\sec^2\theta}\, d\theta}{2\ \cancel{16}\ \sec^{\cancel{4}}\theta} = \int_0^{\arctan(\frac{1}{2})} \frac{\tan^2\theta\, d\theta}{2\sec^2\theta}$$

$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\, d\theta \\ 4+x^2 &= 4+4\tan^2\theta = 4\sec^2\theta \end{aligned}}$$

$$= \int_0^{\arctan(\frac{1}{2})} \frac{1}{2}\sin^2\theta\, d\theta$$

$$\boxed{\cos(2\theta) = 1-2\sin^2\theta}$$

$$= \int_0^{\arctan(\frac{1}{2})} \frac{1}{2}\,\frac{1-\cos(2\theta)}{2}\, d\theta$$

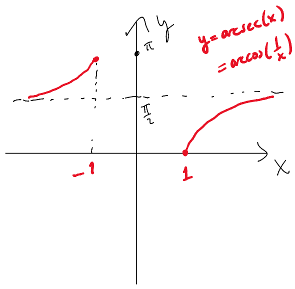
$$= \int_0^{\arctan(\frac{1}{2})} \frac{1}{2}\,\frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \left(\frac{\theta}{4} - \frac{\sin(2\theta)}{8}\right)\Bigg|_0^{\arctan(\frac{1}{2})} = \frac{\arctan(\frac{1}{2})}{4} - \frac{\sin(2\arctan(\frac{1}{2}))}{8}$$

$$= \ldots$$

The arcsec substitution: Suppose that we have $\sqrt{x^2-a^2}$, where $a>0$, in our integrand. (OR, x^2-a^2)

Then we can try to use the substitution



$\theta = \operatorname{arcsec}\left(\frac{x}{a}\right)$

OR

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\,d\theta$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta}$$

$$= |a\tan\theta| = \begin{cases} a\tan\theta & \text{if } x \geq a \\ -a\tan\theta & \text{if } x \leq -a \end{cases}$$

Example: Find $\displaystyle\int_{-2}^{-1} \frac{1}{\sqrt{4x^2-4x-3}}\,dx$

Solution:

$$\int_{-2}^{-1} \frac{1}{\underbrace{\sqrt{4x^2-4x-3}}_{(2x-1)^2-4}}\,dx = \int_{-2}^{-1}\frac{1}{\sqrt{(2x-1)^2-2^2}}\,dx = \int_{\operatorname{arcsec}(\frac{-5}{2})}^{\operatorname{arcsec}(\frac{-3}{2})} \frac{\sec\theta\tan\theta\,d\theta}{\underbrace{\sqrt{4\tan^2\theta}}_{|2\tan\theta|}}$$

$$\boxed{\begin{aligned} 2x-1 &= 2\sec\theta \\ \cancel{2}dx &= \cancel{2}\sec\theta\tan\theta\,d\theta \\ (2x-1)^2-4 &= 4\sec^2\theta - 4 \\ &= 4\tan^2\theta \end{aligned}}$$

$$= \int_{\operatorname{arcsec}(\frac{-5}{2})}^{\operatorname{arcsec}(\frac{-3}{2})} \frac{\sec\theta\,\cancel{\tan\theta}\,d\theta}{-2\,\cancel{\tan\theta}}$$

$\tan\theta < 0$ for $\operatorname{arcsec}(\frac{-5}{2}) < \theta < \operatorname{arcsec}(\frac{-3}{2})$

$$= \frac{-1}{2}\ln|\sec\theta + \tan\theta|\,\Bigg|_{\operatorname{arcsec}(\frac{-5}{2})}^{\operatorname{arcsec}(\frac{-3}{2})} = \ldots$$

The $\tan(\frac{\theta}{2})$ substitution:

$\theta = 2\arctan(x)$

OR

$x = \tan(\frac{\theta}{2})$

$dx = \sec^2(\frac{\theta}{2})\,\frac{1}{2}\,d\theta$

$dx = \left(1+\tan^2(\frac{\theta}{2})\right)\frac{1}{2}\,d\theta$

This substitution may convert an integral involving

The $\tan(\frac{\theta}{2})$ substitution

This substitution may convert an integral involving sine's and cosine's into an integral of a rational function.

OR

$$x = \tan\left(\frac{\theta}{2}\right)$$

$$dx = \left(1+\tan^2\left(\frac{\theta}{2}\right)\right)\frac{1}{2}\,d\theta$$

$$dx = (1+x^2)\,\frac{1}{2}\,d\theta$$

$$\boxed{\cos\theta = \frac{1-x^2}{1+x^2}} \qquad \boxed{\sin\theta = \frac{2x}{1+x^2}} \qquad \boxed{\frac{2\,dx}{1+x^2} = d\theta}$$

Example: Find $\displaystyle\int \frac{1}{2+\sin\theta}\,d\theta$

Solution:

$$\int \frac{1}{2+\sin\theta}\,d\theta = \int \frac{\frac{2}{1+x^2}\,dx}{2+\frac{2x}{1+x^2}} = \int \frac{\frac{\cancel{2}}{\cancel{1+x^2}}}{\frac{\cancel{2}+\cancel{2}x^2+\cancel{2}x}{\cancel{1+x^2}}}\,dx$$

$$\boxed{x = \tan\frac{\theta}{2}}$$

$$= \int \frac{1}{1+x+x^2}\,dx$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \ldots\ldots$$

Use the techniques from 6.2